CASE 1: Intersecting Lines

Find the intersection of \( L_1 \) and \( L_2 \) below:

\[ L_1: r^* = (4,6,9) + s(8,-2,1), s \in R \]
\[ L_2: r^* = (13,-5,-8) + t(3,1,4), t \in R \]

Solution:

\[ m_1^* = (8,-2,1) \text{ and } m_2^* = (3,1,4) \text{ are not collinear. Therefore, the lines either have one point of intersection or they are skew lines.} \]

\begin{align*}
\text{\( L_1 \) parametric equations:} & \quad \text{\( L_2 \) parametric equations:} \\
x &= 4 + 8s & x &= 13 + 3t \\
y &= 6 - 2s & y &= -5 + t \\
z &= 9 + s & z &= -8 + 4t
\end{align*}

Set \( x, y, \text{and } z \) equations together (eliminating \( x, y, \text{and } z \)):

<table>
<thead>
<tr>
<th>( x - \text{equation} )</th>
<th>( y - \text{equations} )</th>
<th>( z - \text{equation} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 + 8s = 13 + 3t</td>
<td>6 - 2s = -5 + t</td>
<td>9 + s = -8 + 4t</td>
</tr>
<tr>
<td>8s - 3t = 9</td>
<td>2s + t = 11</td>
<td>s - 4t = -17</td>
</tr>
</tbody>
</table>

Solve by substitution or elimination:
below is the solution for \( (s,t) \) from the \( y - \text{and } z - \text{equations} \)
from the \( y - \text{equation}: t = 11 - 2s \)
substituting it into the \( z - \text{equation}: \)

\[
\begin{align*}
s - 4t &= -17 \\
4(11 - 2s) - 4s &= -17 \\
9s &= 27 \\
s &= 3
\end{align*}
\]

solve for \( t: \)
\[
\begin{align*}
t &= 11 - 2s \\
&= 11 - 2(3) \\
&= 11 - 6 \\
&= 5
\end{align*}
\]

Check \( (s,t) = (3,5) \) in the \( x - \text{equation}: \)

\[
\begin{align*}
LS &= 8s - 3t \\
&= 8(3) - 3(5) \\
&= 24 - 15 \\
&= 9
\end{align*}
\]

\[
\begin{align*}
RS &= 9 \\
&= LS
\end{align*}
\]

Since \( (s,t) = (3,5) \) satisfies all three equations, we have a point of intersection. We can find it by substituting \( s = 3 \) into \( L_1 \)’s equation or \( t = 5 \) into \( L_2 \)’s equation.

\[
\begin{align*}
L_1: r^* &= (4,6,9) + s(8,-2,1) \\
&= (4,6,9) + 3(8,-2,1) \\
&= (4,6,9) + (24,-6,3) \\
&= (28,0,12)
\end{align*}
\]

Therefore, the intersection of \( L_1 \) and \( L_2 \) is the point \( (28,0,12) \).
CASE 2: Coincident Lines

Find the intersection of $L_1$ and $L_2$ below:

$L_1: \vec{r} = (4,9,2) + s(2,4,-1), s \in R$
$L_2: \vec{r} = (10,21,−1) + t(−4,−8,2), t \in R$

Solution:

$m_2^\perp = (-4,−8,2)$
$m_2^\perp = -2(2,4,1)$
$m_2^\perp = -2m_1^\perp$

Therefore, $m_1^\perp$ and $m_2^\perp$ are collinear and $L_1$ and $L_2$ are parallel.

Is $(10,21,−1)$, the point on $L_2$, also on $L_1$? If so, the lines are parallel and coincident (∞ solutions)
If not, the lines are parallel and distinct (0 solutions)

| Find $s$ so that $(10,21,−1) = (4,9,2) + s(2,4,−1)$ |
|----------------|----------------|----------------|
| 10 = 4 + 2s    | 21 = 9 + 4s    | −1 = 2 − s     |
| 6 = 2s         | 12 = 4s        | −3 = −s        |
| 3 = s          | 3 = s          | 3 = s          |

Since $s = 3$ satisfies each of the parametric equations, we know that $(10,21,−1)$ is on both $L_1$ and $L_2$. Therefore, the lines are coincident and the solution is the whole line given by either equation of $L_1$ or $L_2$. The solution is the line $\vec{r} = (4,9,2) + s(2,4,−1), s \in R$
**CASE 3: Parallel and Distinct Lines**

Find the intersection of $L_1$ and $L_2$ below:

$L_1: r = (5,6,2) + s(3,1,8), s \in R$

$L_2: r = (11,8,16) + t(-9,-3,-24), t \in R$

Solution:

$$m_2^\top = (-9, -3, -24)$$

$$= -3(3,1,8)$$

$$= -3m_1^\top$$

Therefore, $m_1^\top$ and $m_2^\top$ are collinear and $L_1$ and $L_2$ are parallel.

Is $(11,8,16)$ the point on $L_2$ also on $L_1$? If so, the lines are parallel and coincident ($\infty$ solutions)

If not, the lines are parallel and distinct (0 solutions)

Find $s$ so that $(11,8,16) = (5,6,2) + s(3,1,8)$

| 11 = 5 + 3s | 8 = 6 + s | 16 = 2 + 8s |
| 6 = 3s | 2 = s | 14 = 8s |
| 2 = s | 7 | 4 = s |

The $s$ – values are not all the same for each of the three coordinate equations.

Therefore, the point $(11,8,16)$ is not on $L_1$ and the lines are distinct.

Therefore, the lines do not have any intersection.
CASE 4: Skew Lines

Find the intersection of $L_1$ and $L_2$ below:

$L_1: r = (8,2,1) + s(4,3,6), s \in \mathbb{R}$

$L_2: r = (10,-5,10) + t(2,4,1), t \in \mathbb{R}$

Solution:

$m_1 = (4,3,6)$ and $m_2 = (2,4,1)$ are not collinear. Therefore, the lines either have one point of intersection or they are skew lines.

<table>
<thead>
<tr>
<th>$L_1$ parametric equations:</th>
<th>$L_2$ parametric equations:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 8 + 4s$</td>
<td>$x = 10 + 2t$</td>
</tr>
<tr>
<td>$y = 2 + 3s$</td>
<td>$y = -5 + 4t$</td>
</tr>
<tr>
<td>$z = 1 + 6s$</td>
<td>$z = 10 + t$</td>
</tr>
</tbody>
</table>

Set $x, y,$ and $z$ equations together (eliminating $x, y,$ and $z$):

<table>
<thead>
<tr>
<th>$x$ – equation</th>
<th>$y$ – equations</th>
<th>$z$ – equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$8 + 4s = 10 + 2t$</td>
<td>$2 + 3s = -5 + 4t$</td>
<td>$1 + 6s = 10 + t$</td>
</tr>
<tr>
<td>$4s - 2t = 2$</td>
<td>$3s - 4t = -8$</td>
<td>$6s - t = 9$</td>
</tr>
</tbody>
</table>

Solve by substitution or elimination:

below is the solution for $(s, t)$ from the $x$ – and $z$ – equations

from the $z$ – equation: $t = 6s - 9$

substituting it into the $x$ – equation:

$solve for t:$

$t = 6s - 9$

$t = 6(2) - 9$

$t = 12 - 9$

$t = 3$

Check $(s, t) = (2,3)$ in $y$ – equation:

$LS = 3s - 4t$

$= 3(2) - 4(3)$

$= 6 - 12$

$= -6$

$RS = -8$

$\neq LS$

Since $(s, t) = (2,3)$ only works in two equations and not the third, there is no point of intersection. Therefore, the lines $L_1$ and $L_2$ are skew and there is no intersection.